GEOMECHANICS

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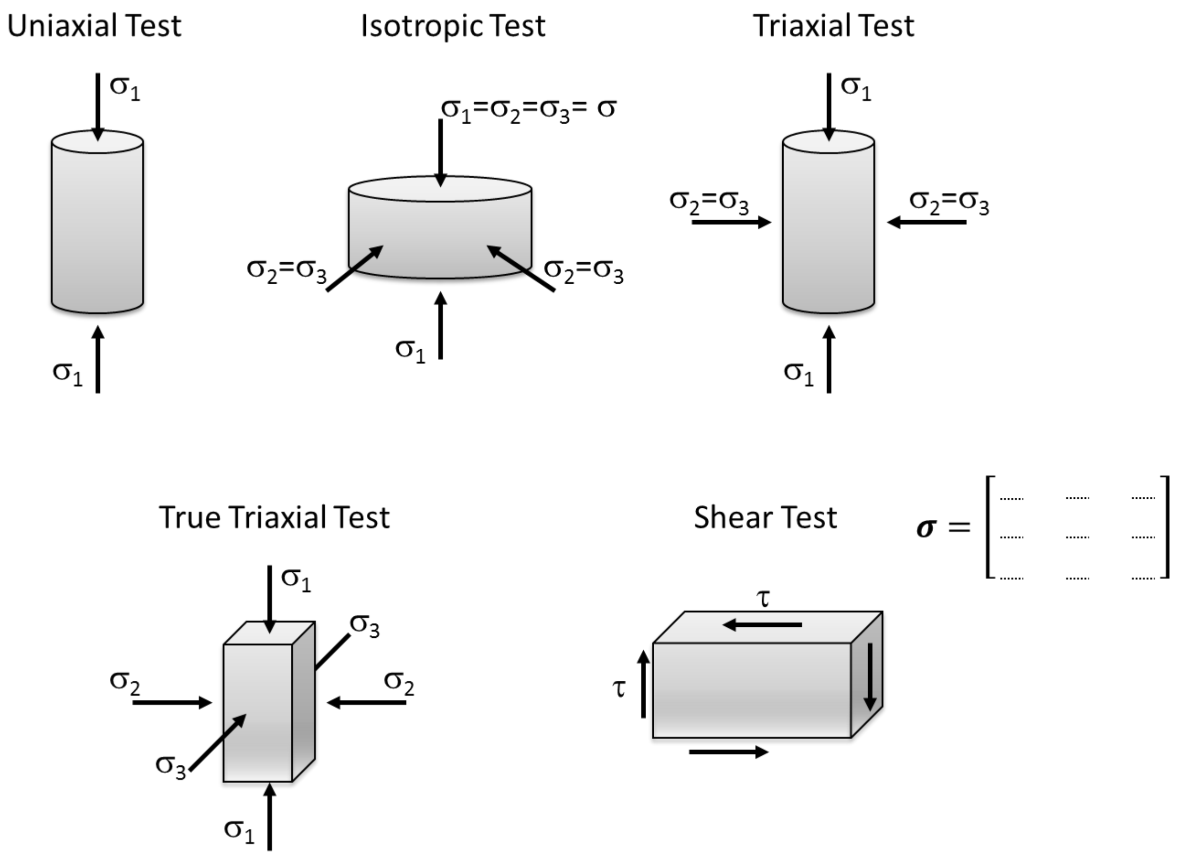
Exercise 2 – 02.10.2025 – Solution

Elasticity

1. Stress tensors definitions

Part 1: Qualitative cases

Define the stress tensor for the following cases representative of the most common laboratory test conditions:



Uniaxial test

$$\boldsymbol{\sigma_{ij}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_1 \end{bmatrix}$$

Isotropic test

$$\boldsymbol{\sigma_{ij}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

Triaxial test

$$\boldsymbol{\sigma_{ij}} = \begin{bmatrix} \sigma_3 & 0 & 0 \\ 0 & \sigma_3 & 0 \\ 0 & 0 & \sigma_1 \end{bmatrix}$$

True triaxial test

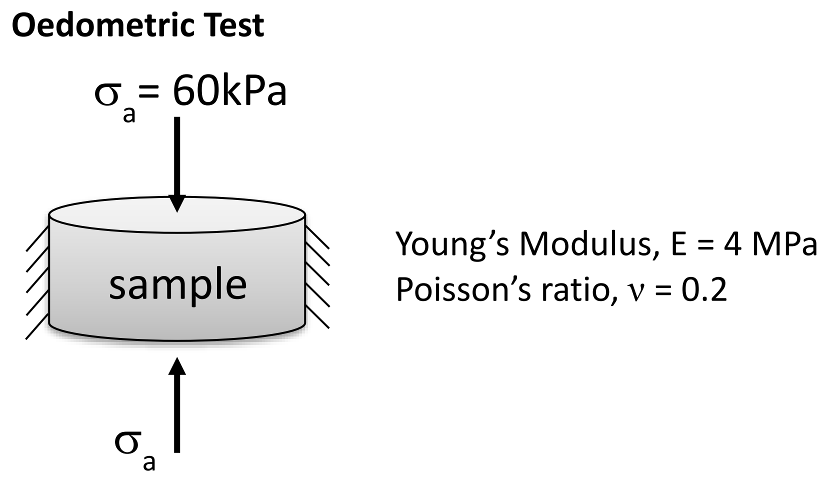
$$\boldsymbol{\sigma_{ij}} = \begin{bmatrix} \sigma_3 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1 \end{bmatrix}$$

Shear test

$$\boldsymbol{\sigma_{ij}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}$$

Part 2: Computation example

Consider an oedometric test where the sample is subjected to a vertical axial stress of 60 kPa. Given the properties of the tested material, calculate the different stress components of the stress tensor. (Use the elastic relationship between the stress and strain components)



General elastic relationship for normal strain-normal stress relationship:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

In oedometric conditions, $\varepsilon_x = \varepsilon_y = \varepsilon_r = 0$, $\sigma_x = \sigma_y = \sigma_r$, and $\sigma_z = \sigma_v$

Substituting these conditions in the three equations above, we can obtain:

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_r + \sigma_v)\right] = 0$$

Which gives:

$$\sigma_r = \frac{\nu}{1-\nu}\sigma_v = 15\ kPa$$

Stress tensor:

$$\boldsymbol{\sigma_{ij}} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 60 \end{bmatrix}$$

2. Parameter determination for an undrained elastic material

A conventional undrained triaxial compression test, with the cell pressure σ_c, held constant, is carried out on a sample of stiff overconsolidated clay. The stress-strain relationship is found to be linear up to failure, so it is deduced that the clay behaves as an isotropic elastic material. The Biot coefficient can be considered to be 1 (due to incompressible grains). Consider a back pressure (initial pore water pressure) equal to 0. ***Tip:*** *to compute G, consider the following equation $\Delta q = 2G(\Delta\varepsilon_{11} - \Delta\varepsilon_{33})$ obtained through the relationship between deviatoric stress and invariants (derivation annexed).*

Part 1

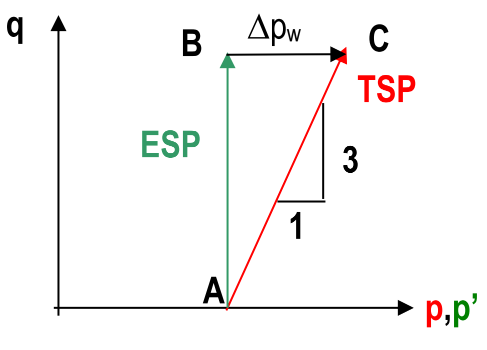
After an axial strain $\Delta\varepsilon_a$ = 0.8%, the corresponding deviatoric stress is q = Δq = 80*kPa*.

Calculate the corresponding values of:

- Mean total stress Δp,
- Mean effective stress $\Delta p'$
- Pore water pressure Δp_w
- Radial effective stress $\Delta\sigma'_r$
- Axial total stress $\Delta\sigma_a$
- Axial effective stress $\Delta\sigma'_a$
- Radial deformation $\Delta\varepsilon_r$
- Undrained Young's modulus E_u
- Undrained shear modulus G

After an axial strain $\Delta\varepsilon_a$ = 0.8%, q = 80*kPa*.

In the stress plane $(q - p')$ (effective stress path - ESP) and $(q - p)$ (Total stress path-TSP) the representation of loading path is the following



Notice that in the fully drained case, the ESP would be identical to the TSP.

The corresponding ESP is a straight vertical line in this case because the tested soil is assumed to be an ideal isotropic elastic material, i.e. its Skempton parameter $A = 1/3$.

$\Delta p' = \Delta p - \Delta p_w$ with,

$\Delta p_w = \Delta\sigma_3 + \frac{1}{3}(\Delta\sigma_1 - \Delta\sigma_3)$ → $\Delta p_w = \frac{1}{3}(\Delta\sigma_1 + 2\Delta\sigma_3)$ → $\Delta p_w = \Delta p$

Here, the undrained condition implies:

- **Mean total stress:** the variation in total stress is related to the variation in deviatoric stress and the CTC slope (3) in the $(q - p)$ plane → $\Delta p = \frac{\Delta q}{3} = 26.7kPa$
- **Mean effective stress:** as no volumetric deformations are experienced by the sample due to the undrained condition → $\Delta\varepsilon_v = 0 = \frac{\Delta p'}{K} \rightarrow \Delta p' = 0$
- **Pore water pressure:** the difference between the total stress and the effective stress being the pore pressure (Terzaghi's definition of effective stress) → $\Delta p_w = \Delta p - \Delta p' = \Delta p = 26.7kPa$
- **Radial effective stress:** the definition of the effective stress is applicable to all stresses
  → $\Delta\sigma_r' = \Delta\sigma_r - \Delta p_w = 0 - 26.7 = -26.7kPa$
- **Axial total stress:** using the definition of the deviatoric stress
  → $\Delta q = \Delta\sigma_a - \Delta\sigma_r \rightarrow \Delta\sigma_a = \Delta q + 0 = 80kPa,$
- **Axial effective stress:** → $\Delta\sigma_a' = \Delta\sigma_a - \Delta p_w = 53.3kPa$
- **Radial deformation:** using the definition of the volumetric strain:
  → $\Delta\varepsilon_v = 0 = \Delta\varepsilon_a + 2\Delta\varepsilon_r \rightarrow \Delta\varepsilon_r = -\frac{0.008}{2} = -0.004$
- **Undrained Young's modulus** → $E_u = \frac{\Delta q}{\Delta\varepsilon_a} = 10^4 kPa$
- **Undrained shear modulus**
  →The deviatoric stress q is related to the invariant I_{2D} by the shear modulus G, according to the following equation:
  $q = 2\sqrt{3}G\sqrt{I_{2D}}$

  →This equation is obtained from the relationship between deviatoric stress q and the second invariant J_{2D} of the deviatoric stress tensor s_{ij}

  $$J_{2D} = \frac{1}{6}[(\sigma_1 - \sigma_3)^2 + (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2] = \frac{1}{6}[2(\sigma_1 - \sigma_3)^2] = \frac{1}{3}[(\sigma_a - \sigma_r)^2]$$
  $$\mathrm{q} = \sqrt{3\mathrm{J}_{2\mathrm{D}}}$$

  →Considering the elastic constitutive relationship between deviatoric stress tensor s_{ij} and deviatoric strain tensor e_{ij}
  $s_{ij} = 2Ge_{ij}$

  →J_{2D} can be written as
  $\sqrt{\mathrm{J}_{2\mathrm{D}}} = 2\mathrm{G}\sqrt{\mathrm{I}_{2\mathrm{D}}}$
  where, in triaxial conditions, I_{2D} is second invariant of the deviatoric strain tensor e_{ij}
  $$\mathrm{I}_{2\mathrm{D}} = \frac{1}{6}[(\varepsilon_1 - \varepsilon_3)^2 + (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2] = \cdots = \frac{1}{3}(\varepsilon_a - \varepsilon_\mathrm{r})^2$$

  →Finally, we can write $q = 2G(\varepsilon_{11} - \varepsilon_{33})$, and $\Delta q = 2G(\Delta\varepsilon_{11} - \Delta\varepsilon_{33})$
  →Along path AC: $\Delta\sigma_a - \Delta\sigma_r = 2G(\Delta\varepsilon_a - \Delta\varepsilon_r) \rightarrow G = 3333kPa$

Part 2

At this time, the axial stress and cell pressure are kept constant, and the sample is allowed to drain so that the pore pressures dissipate and the sample undergoes a volumetric strain $\Delta\varepsilon_v$ = 0.25%

Calculate the values of $\Delta p_w, \Delta\sigma'_a, \Delta\sigma'_r, \Delta q, \Delta p', \Delta\varepsilon_a,\ \Delta\varepsilon_r$

Calculate the value of bulk modulus K.

Drainage under constant axial stress and cell; the path corresponds to 'BC' in the figure.

Volumetric strain $\Delta\varepsilon_v$ = 0.25%.

- Pore pressure dissipate, so at the end of the drainage process it has to be equal to zero →$\Delta p_w = 0 - 26.7 = -26.7kPa$
- $\Delta\sigma'_a = \Delta\sigma_a - \Delta p_w = 0 - (-26.7) = 26.7kPa$
- $\Delta\sigma'_r = \Delta\sigma_r - \Delta p_w = 0 - (-26.7) = 26.7kPa$
- $\Delta q = \Delta\sigma_a - \Delta\sigma_r = 0$ kPa
- $\Delta p' = \frac{\Delta\sigma'_a + 2\Delta\sigma'_r}{3} = 26.7kPa$
- Along path BC: $\Delta\varepsilon_v = \frac{1}{K}\Delta p' \rightarrow K = 10680kPa$

Note that pore pressure variation produces the same effective stress in both axial and radial directions; as a consequence, only variation of mean effective stress is observed while deviatoric stress remains constant.

- Using again the following equation $q = 2G(\varepsilon_a - \varepsilon_r)$, and considering that along the path BC $\Delta q = 0$, we can write:

  $\Delta\sigma_a - \Delta\sigma_r = 2G(\Delta\varepsilon_a - \Delta\varepsilon_r)$
  $0 = 2G(\Delta\varepsilon_a - \Delta\varepsilon_r)$

  then $\Delta\varepsilon_a = \Delta\varepsilon_r$

  Furthermore, we can also write:

$$0 = 2G(\Delta\varepsilon_a - \Delta\varepsilon_r) = 2G\left(\Delta\varepsilon_a - \frac{\Delta\varepsilon_v - \Delta\varepsilon_a}{2}\right) = G(3\Delta\varepsilon_a - \Delta\varepsilon_v)$$

where $\Delta\varepsilon_v = \Delta\varepsilon_a + 2\Delta\varepsilon_r$

→$\Delta\varepsilon_a = \frac{1}{3}\Delta\varepsilon_v = 0.00083$
→$\Delta\varepsilon_r = \Delta\varepsilon_a = 0.00083$